On the other hand, a similar field matrix U_N may be obtained for the subregion N which tends to infinity together with $y \to \infty$

$$H_N|_{y_N} = U_N \cdot E_N|_{y_N}. \quad (10)$$

Matching the tangential field components of subregions $N-1$ and N at $y = y_N$, a characteristic matrix is obtained and frequency-dependent propagation constants and related field parameters can be found by solving this determinant equation.

C. Numerical Results

To validate the proposed recursive algorithm, the characteristic impedance of even and odd modes of a coupled microstrip line is calculated by using the quasistatic modeling, showing a good agreement with [11]. Fig. 2 shows the capacitance matrix of a three-conductor microstrip line. Fig. 3 displays the capacitance matrix of a four-conductor microstrip line deposited on a segmented multilayer uniaxial anisotropic substrate. A typical CPU time is 5 min. for the calculation of this figure on a low-speed HP-400 workstation. The limiting line spacing used is around 300/mm. The calculated results change significantly with the normalized height of the segmented layer t_1. It is interesting that the self-capacitances C_{11}, C_{44}, and C_{33} tend to equal each other when the thickness t_1 approaches zero and their values diversify as t_1 becomes large. This can be explained by the fact that when $t_1 \to 0$, the coupling between the different strips increases drastically and the coupling effect makes C_{33} and C_{44} increase faster than C_{11} since the dimension s_3 is much less than s_1. As the thickness t_1 increases, the coupling effect diminishes and the difference between the self-capacitance become more pronounced.

Dispersion characteristics of a coupled microstrip line are also calculated. Fig. 3 shows dispersion characteristics of different modes of a microstrip line with four conductors on a segmented multilayered substrate.

III. Conclusion

This paper presents a recursive algorithm of the method of lines based on the vertical discretization [1] for the analysis of multiple strips or slots on composite multilayered substrates including uniaxial anisotropic materials. The advantage of this algorithm is that modeling on arbitrary multiple lines (or slots) is accomplished by a simple transferring process of the "standard" field matrices from one strip (or slot) to another. An additional identified advantage compared to the conventional method of lines is that the order of characteristic matrix remains always the same regardless of the number of strips or slots. This is more pronounced when a large number of strips or slots gets involved such as in high-speed interconnects. Our examples demonstrate potential application to a large class of planar circuits including complex composite substrates with hollow segments which were proposed to reduce the field coupling between different strips.

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The Propagation Constant of a Lossy Coaxial Line with a Thick Outer Conductor

W. C. Daywitt

Abstract—**The microwave approximation for the propagation constant of a coaxial line becomes inaccurate below 1 MHz. An approximation is presented that is accurate over the entire operating frequency range of the line.**

I. Introduction

The propagation constant γ for the principal, transverse magnetic mode on a lossy coaxial line has been known for many years [1] and appears in the field equations with the form

$$\boldsymbol{F} = \boldsymbol{F}_0 e^{j\omega t - \gamma z} \quad (1)$$

where $\boldsymbol{F}$ represents any one of the principal mode field components, ω the radian frequency, t the time, and z the axial distance along the line.

An exact calculation of γ is complicated by the need to solve the coaxial line determinantal equation involving Bessel functions of the first and second kinds with complex arguments. Furthermore, the only approximation for γ in common usage today is that originally derived by Stratton [1], a first-order perturbation equation in the

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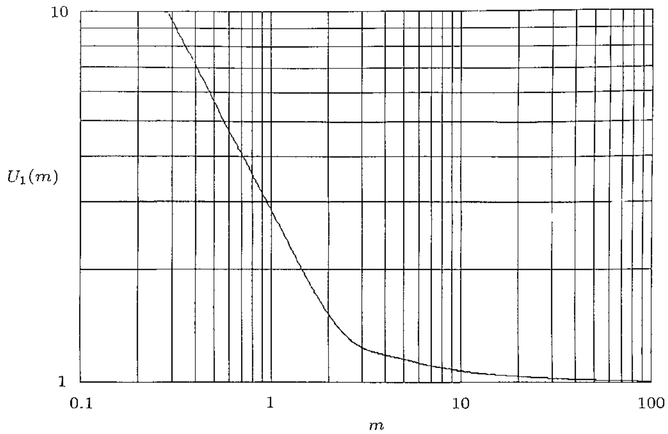

Fig. 1. Inner conductor correction coefficient $U_1(m)$ for the full range approximation to the attenuation coefficient.

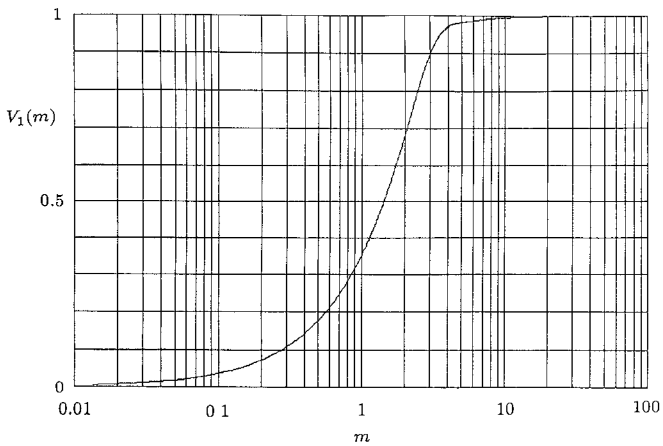

Fig. 2. Outer conductor correction coefficient $V_1(m)$ for the full range approximation to the attenuation coefficient.

conductor skin depth or surface impedance, which is accurate only in the microwave part of the line's useful frequency range.

In a 7-mm copper line, for example, this microwave approximation degrades significantly as the operating frequency falls below 1 MHz. Fortunately, the magnitudes of the real and imaginary parts of the line loss corrections to the propagation constant diminish rapidly with frequency so that even a large relative error in the corrections is unimportant for many practical applications. This is not always the case, however, especially in the field of metrology where sound error analyses [3] are based upon accurate mathematical descriptions.

A fairly recent development [2][1] using Stratton's 1941 equations outlines a procedure whereby a full frequency range approximation to the propagation constant can be obtained without too much additional calculational effort beyond what is required for the microwave approximation mentioned above. This procedure has been implemented and the results are presented in what follows.

II. A Full Range Approximation for γ

The propagation constant for a lossy coaxial line with an infinitely thick outer conductor can be calculated from the following equation derived from the results in [2]

$$\gamma = jk_2(1 + \epsilon_1 - j\epsilon_2)^{1/2}.$$

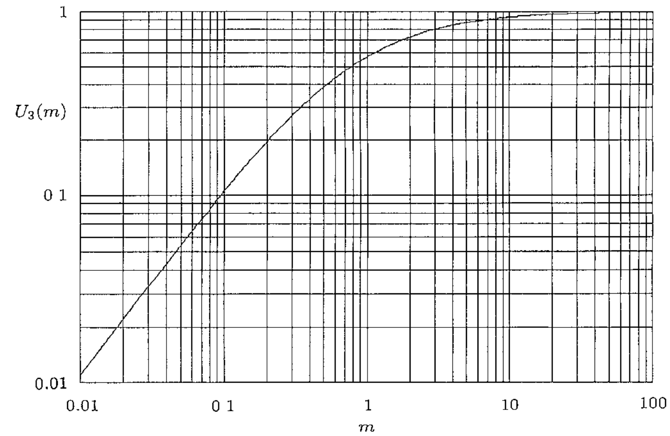

Fig. 3. Inner conductor correction coefficient $U_3(m)$ for the full range approximation to the phase coefficient.

$$\epsilon_1 \equiv \frac{1}{\ln b/a}\left(\frac{V_1\delta_1}{2a} + \frac{V_3\delta_3}{2b}\right),$$
$$\epsilon_2 \equiv \frac{1}{\ln b/a}\left(\frac{U_1\delta_1}{2a} + \frac{U_3\delta_3}{2b}\right) \quad (2)$$

where

$$\delta_i = \left(\frac{2\rho_i}{\omega\mu}\right)^{1/2} \quad (3)$$

is the inner ($i = 1$) or outer ($i = 3$) conductor skin depth. The constant k_2 is the wavenumber within the dielectric (assumed to be lossless for convenience) occupying region 2 between the conductors; μ is the permeability of free space; a and b are the radii of the inner conductor (region 1) and outer conductor (region 3) respectively; ρ_1 and ρ_3 are the resistivities for the inner and outer conductors; and U_1, V_1, U_3, and V_3 are the functions discussed below. Equation (2) is accurate over the full frequency range of the line (e.g., 0 to 18 GHz for a 7 mm transmission line).

Writing γ as $\alpha + j\beta$ leads to

$$\alpha = \frac{k_2\epsilon_2}{2^{1/2}\{1 + \epsilon_1 + [(1+\epsilon_1)^2 + \epsilon_2^2]^{1/2}\}^{1/2}} \quad (4)$$

and

$$\beta = k_2\left\{\frac{1 + \epsilon_1 + [(1+\epsilon_1)^2 + \epsilon_2^2]^{1/2}}{2}\right\}^{1/2} \quad (5)$$

where α and β are the line's attenuation and phase coefficients. The usual microwave approximation to these coefficients is obtained by expanding (4) and (5) to first order in the ϵ_i ($i = 1, 2$) while setting U_1, V_1, U_3, and V_3 in (2) to unity.

The U and V functions are obtained from the R_1 and R_3 functions in [2] by setting

$$U_1(m_1) + jV_1(m_1) = \frac{-j(1+j)}{R_1(m_1 e^{-j\pi/4})} \quad (6)$$

and

$$U_3(m_3) + jV_3(m_3) = \frac{j(1+j)}{R_3(m_3 e^{-j\pi/4})} \quad (7)$$

where (i = 1, 3)

$$m_i \equiv \frac{2^{1/2} r_i}{\delta_i} \quad (8)$$

and where $r_1 = a$ and $r_3 = b$. Plots of the U and V functions versus m are shown in Figs. 1–4.

[1] Equation (38) in this paper is incorrect and should be replaced by $\hat{\gamma} = jk_2(1 - \hat{h}^2/k_2^2)^{1/2}$. Equations (40)–(45) should be discarded. The remaining equations and the results quoted are correct.

TABLE I
POLYNOMIAL APPROXIMATION FOR $U_1(m)$

Range	Approximation
$0 \le m \le 0.06$	$U_1 = \frac{2^{3/2}}{m}$
$0.06 < m \le 0.5$	$U_1 = 0.00717907 + \frac{2.82409}{m} + \frac{0.000998}{m^2} - \frac{0.000107}{m^3} + \frac{0.0000054}{m^4} - \frac{0.0000001044}{m^5}$
$0.5 < m \le 1.5$	$U_1 = 0.339869 + \frac{2.01144}{m} + \frac{0.75945}{m^2} - \frac{0.31705}{m^3} + \frac{0.0496566}{m^4}$
$1.5 < m < 5$	$U_1 = 7.843851 - 9.28113m + 6.162m^2 - 2.522427m^3 + 0.67249018m^4 - 0.11313104m^5 + 0.010809984m^6 - 0.00044425172m^7$
$5 \le m < 10$	$U_1 = 1.183935 + 0.1010435m - 0.04749808m^2 + 0.007559972m^3 - 0.000538619m^4 + 0.00001456081m^5$
$10 \le m < 200$	$U_1 = 0.9999853 + \frac{0.708036}{m} + \frac{0.36129}{m^2}$
$200 \le m$	$U_1 = 1 + \frac{1}{2^{1/2}m}$

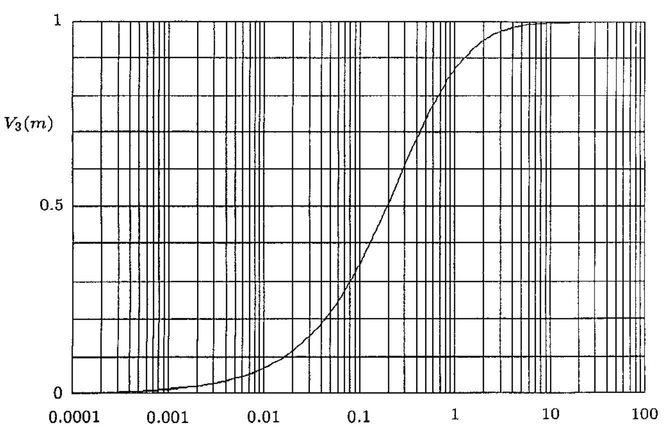

Fig. 4. Outer conductor correction coefficient $V_3(m)$ for the full range approximation to the phase coefficient.

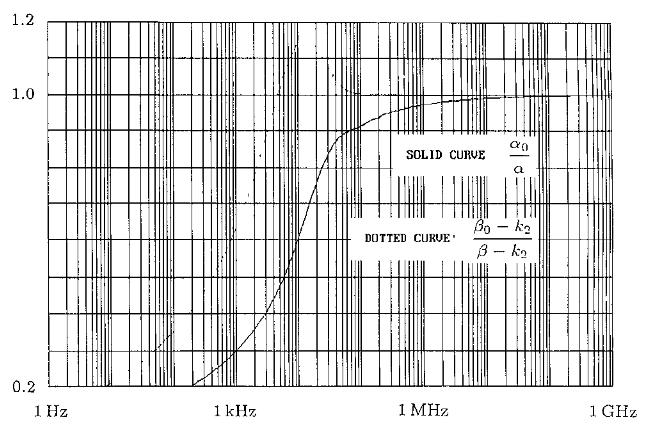

Fig. 5. The ratios of the microwave to full range approximations of the attenuation and phase coefficients for a 7-mm line. The calculations assumed a resistivity of 2 microhm-centimeters.

The $R_1(\,)$ and $R_3(\,)$ appearing in (6) and (7) are related to ratios of first- and third-kind Bessel functions of the complex argument appearing in the above $R(\,)$s and are a nuisance to calculate. As pointed out in [2], however, (6) and (7) can be accurately represented by polynomial approximations. Such approximations were derived by regression fitting polynomials to data generated from the exact functions given in [2] and are presented in Tables I–IV, where the approximations are accurate to better than 1 part in 10^4 or 0.01%. The first and last entries in each table are the leading terms of the small argument and asymptotic expansions for the Us and Vs.

TABLE II
POLYNOMIAL APPROXIMATION FOR $V_1(m)$

$0 \le m \le 0.1$
$V_1 = \frac{m}{2^{3/2}}$
$0.1 < m < 1$
$V_1 = -0.000014 + 0.353774m - 0.001119m^2 + 0.0024874m^3 - 0.00249m^4$
$1 \le m \le 4$
$V_1 = -0.066604 + 0.57411785m - 0.29303277m^2 + 0.19966757m^3 - 0.0728026m^4 + 0.01203129m^5 - 0.00074024333m^6$
$4 < m \le 8$
$V_1 = 0.7867 + \frac{5.66171}{m} - \frac{56.96857}{m^2} + \frac{259.1723}{m^3} - \frac{522.158}{m^4} + \frac{326.783}{m^5}$
$8 < m < 50$
$V_1 = 1.000021 - \frac{0.00153}{m} - \frac{0.342954}{m^2} - \frac{0.7592}{m^3}$
$50 \le m$
$V_1 = 1 - \frac{3}{8m^2}$

TABLE III
POLYNOMIAL APPROXIMATION FOR $U_3(m)$

$0 \le m \le 0.01$
$U_3 = \frac{\pi m}{2^{3/2}}$
$0.01 < m \le 0.1$
$U_3 = -0.000009454 + 1.11299m - 0.26823m^2 - 5.955m^3 + 34.02m^4 - 82m^5$
$0.1 < m \le 1$
$U_3 = -0.0013002 + 1.156549m - 0.934815m^2 + 0.1536m^3 + 0.82777m^4 - 1.1821m^5 + 0.71657m^6 - 0.17026m^7$
$1 < m < 100$
$U_3 = 1.00003 - \frac{0.7089912}{m} + \frac{0.401145}{m^2} - \frac{0.1544}{m^3} + \frac{0.028238}{m^4}$
$100 \le m$
$U_3 = 1 - \frac{1}{2^{1/2}m}$

III. DISCUSSION

The expressions for ϵ_1 and ϵ_2 in (2) lead to accuracies in (4) and (5) that are better than 1 part in 10^5 for α and 1 part in 10^{13} for β ($\rho_1 = \rho_3 = 2$ microhm-cm) at the upper frequency limit of the line (18 GHz in 7mm line) with the approximations becoming rapidly more accurate as the frequency decreases below this limit (see Fig.

TABLE IV
POLYNOMIAL APPROXIMATION FOR $V_3(m)$

0 ≤ m ≤ 0.001
$V_3 = \frac{2^{1/2} m \log_{10}[2/m \exp(0.5772156649)]}{\log_{10}(e)}$
0.001 < m ≤ 0.1 $x \equiv m \log_{10}(m)$
$V_3 = -0.00001499 - 3.310182x + 2.0287x^2 + 45.49x^3 + 1137.7x^4 + 17285x^5 + 166160x^6 + 879300x^7 + 2024000x^8$
0.1 < m ≤ 0.3
$V_3 = 0.063389134 + 3.7351412m - 11.108569m^2 + 21.167468m^3 - 17.945046m^4$
0.3 < m < 1
$V_3 = 1.035053 - \frac{0.1636453}{m} - \frac{0.01473093}{m^2} + \frac{0.018285424}{m^3} - \frac{0.004462092}{m^4} + \frac{0.0003841}{m^5}$
1 ≤ m ≤ 40
$V_3 = 1.00003 - \frac{0.0013935}{m} - \frac{0.3573739}{m^2} + \frac{0.4388015}{m^3} - \frac{0.2972413}{m^4} + \frac{0.088049}{m^5}$
40 < m
$V_3 = 1 - \frac{3}{8m^2}$

8 in [2]). Furthermore, the polynomial approximations for the Us and Vs found in Tables I–IV are accurate to better than 1 part in 10^4. Therefore, the U and V errors dominate and (4) and (5) can be counted on to be accurate to better than 1 part in 10^4 over the full, usable, transmission line frequency range. These errors are scaled by $(\rho/2)^{1/2}$ for conductor resistivities other than 2 microhm-cm.

The microwave approximations, α_0 and β_0, for the attenuation and phase coefficients are obtained from (4) and (5) by setting the U and V functions in (2) to unity and expanding (4) and (5) to first order in the ϵ_i

$$\alpha_0 = \frac{k_2 \epsilon_2}{2} \tag{9}$$

and

$$\beta_0 = k_2\left(1 + \frac{\epsilon_1}{2}\right) \tag{10}$$

where

$$\epsilon_1 = \epsilon_2 = \frac{1}{\ln b/a}\left(\frac{\delta_1}{2a} + \frac{\delta_3}{2b}\right). \tag{11}$$

The full-range and microwave approximations are compared in Fig. 5, where the solid curve is the ratio α_0/α and the dotted curve is the ratio $(\beta_0 - k_2)/(\beta - k_2)$.

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